Delayed Twin Observables

Are They a Fundamental Concept in Quantum Mechanics?

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Opposite-subsystem twin events and twin observables, studied previously in the context of distant correlations, are first generalized to pure states of not-necessarily-composite systems, and afterwards they are further generalized to delayed twins that are due to unitary evolution of the quantum system. The versatile aspects of delayed twin observables are studied in terms of necessary and sufficient conditions to make possible various applications. Three of these are sketched: Preparation of some quantum experiments, easy solution of a puzzle in an important Scully et al. real experiment, and, finally, it is shown that exact measurement in quantum mechanics is an example of opposite-subsystem delayed twins in bipartite pure states.

1 INTRODUCTION

Opposite-subsystem twin observables in bipartite systems were previously studied in a number of articles [1], [2], [3], [4]. To illustrate subsystem twins, we consider the well known singlet state

$$|\Psi\rangle_{AB} \equiv (1/2)^{1/2}\Big(\, |+\rangle_A \, |-\rangle_B - \, |-\rangle_A \, |+\rangle_B \Big).$$

The plus and minus stand for the spin-up and spin-down respectively along the z-axis. The twin observables (operators) in this bipartite

state are the z-projections: $\quad s^z_A \quad$ and $\quad -s^z_B$. They are twins because they act equally on the composite state vector $\quad | \Psi \rangle_{AB} \quad$ (cf Definitions 6 and 2 below).

The singlet state vector is written in the form of a twin-adapted canonical Schmidt (or bi-orthogonal) expansion. "Twin-adapted" means that the subsystem state vectors appearing in the expansion are eigen-vectors of the twin observables; "canonical" means that the squares of the expansion coefficients are the eigenvalues of the reduced density operators (subsystem states). The mentioned properties of the expansion of the singlet state vector are general in the sense that every state vector of a bipartite system can be written as a twin-adapted canonical Schmidt expansion (cf section 2 in [5]).

The most important properties of twin observables are that they have the same probabilities in the state at issue, and that they give, by ideal measurement, the same change of the composite-system state.

In this article the concept of twin observables is first generalized to systems that are not necessarily composite. Events (projectors), as the simplest observables, are studied in detail, and the results are utilized for general observables (but restricted to those that do not have a continuous part in their spectrum). The main purpose of this generalization is that it is the basis for further generalization.

The next generalization is to delayed twins, in which a unitary evolution operator is allowed to separate the twin observables.

The last section is devoted to three applications.

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2 FIRST GENERALIZATION: TWIN EVENTS IN PURE STATES OF NOT-NECESSARILY-COMPOSITE SYSTEMS

The terms 'event' and 'projector', 'observable' and 'Hermitian operator', as well as 'pure state' and state vector (vector of norm one) are used interchangeably throughout. If E denotes an event, then E^c denotes the opposite event (the ortho-complementary or simply complementary projector) $E^c \equiv I - E$, where I is the identity operator. The reader should be reminded that the probability of an event E in a state $|\psi\rangle$ is given by the expression $\langle \psi | E | \psi \rangle$ in quantum mechanics.

We are going to need a known but perhaps not well known lemma.

Lemma. An event (projector) P is **certain** (has probability one) in a pure state $|\phi\rangle$ if and only if $|\phi\rangle = P |\phi\rangle$ is valid.

Proof. The claim is almost obvious if one has in mind the identities $|\phi\rangle = P |\phi\rangle + P^c |\phi\rangle$ and $1 = || \, |\phi\rangle ||^2 = ||P \, |\phi\rangle ||^2 + ||P^c \, |\phi\rangle ||^2$. □

We will also need the following simple fact. If E is an event and $|\psi\rangle$ a state vector, then the latter can be viewed as consisting of two **coherent** (cf the Introduction in [6]) component state vectors, each having a sharp value 1 or 0 of the observable $E = 1E + 0E^c$:

$$|\psi\rangle = ||E \, |\psi\rangle || \times \Big(E \, |\psi\rangle \Big/ ||E \, |\psi\rangle || \Big) + ||E^c \, |\psi\rangle || \times \Big(E^c \, |\psi\rangle \Big/ ||E^c \, |\psi\rangle || \Big). \tag{1}$$

Definition 1. If an event E has, as an observable, the sharp

value 1 in a state $|\psi\rangle$, then we say that E 'has occurred' in the latter; if E has the sharp value 0, then the opposite event E^c has occurred in it. If a state $|\psi\rangle$ changes into its first component in decomposition (1), a state in which E has occurred, then we say that E **occurs ideally** or 'in an ideal way' or 'in ideal measurement' in $|\psi\rangle$. (We are dealing with the selective, pure-state Lüders change-of-state formula [7], [8], [9]).

Definition 2. Let E and F be two events and $|\psi\rangle$ a state vector. If the projectors act equally on the vector

$$E\,|\psi\rangle = F\,|\psi\rangle, \tag{2}$$

then we have **twin events** or events that are twins.

The following theorems present twin events in a versatility of mathematical forms and **physical meanings**.

Theorem 1. Two (simple and composite) necessary and sufficient conditions - alternative definitions - (i) and (ii) for two events E and F to be **twins** in a pure state $|\psi\rangle$ read:

(i) The opposite events are twins in $|\psi\rangle$:

$$E^c\,|\psi\rangle = F^c\,|\psi\rangle. \tag{3}$$

(ii) (a) The events E and F have the **same probability** in $|\psi\rangle$:

$$\langle\psi|\,E\,|\psi\rangle = \langle\psi|\,F\,|\psi\rangle. \tag{4a}$$

(ii) (b) If the probability (given by (4a)) is positive, then the two events bring about the **same change of the state in ideal occurrence**, i. e., $|\psi\rangle$ changes into:

$$E\,|\psi\rangle\big/\|E\,|\psi\rangle\| = F\,|\psi\rangle\big/\|F\,|\psi\rangle\|. \tag{4b}$$

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Proof. (i) Subtracting (2) from $I \mid \psi\rangle = I \mid \psi\rangle$, one obtains condition (3). Since (2) and (3) are in a symmetrical relation, also the sufficiency of (3) for (2) is proved.

(ii) Condition (ii) (a) follows obviously from (2), and, if the probabilities in (4a) are positive, so does (ii) (b). Conversely, if (ii) (a) is valid and the probabilities in (4a) are positive, then the denominators in (ii) (b) are equal, then (ii) (b), if valid, it makes also the nominators equal. Thus (2) is valid.

If, on the other hand, the probabilities in (4a) are zero, which amounts to $||E \mid \psi\rangle|| = ||F \mid \psi\rangle|| = 0$, then (2) is obviously implied in the form $E \mid \psi\rangle = F \mid \psi\rangle = 0$. $\qquad\qquad\square$

The conditions (i) and (ii) in Theorem 1 cover both extremes: $0 = E \mid \psi\rangle = F \mid \psi\rangle$ and $E \mid \psi\rangle = F \mid \psi\rangle = \mid \psi\rangle$ of twin projectors. In many concrete cases the first extreme is irrelevant. Now we exclude it, and give two more conditions.

Theorem 2. Let E and F be two events and $\mid \psi\rangle$ a pure state such that both events have positive probability in it. Then, two (composite) necessary and sufficient conditions - alternative definitions - (i) and (ii) for twin events are:

(i) (a) F is a certain event in the state $E \mid \psi\rangle \big/ ||E \mid \psi\rangle||$ (which is the first component in (1)): $E \mid \psi\rangle \big/ ||E \mid \psi\rangle|| = F\big(E \mid \psi\rangle \big/ ||E \mid \psi\rangle||\big)$.

(i) (b) *Vice versa*, E is a certain event in the state $F \mid \psi\rangle \big/ ||F \mid \psi\rangle||$: $F \mid \psi\rangle \big/ ||F \mid \psi\rangle|| = E\big(F \mid \psi\rangle \big/ ||F \mid \psi\rangle||\big)$. This state is a component

in the coherent state decomposition

$$|\psi\rangle = ||F\,|\psi\rangle|| \times \left(F\,|\psi\rangle\big/||F\,|\psi\rangle||\right) + ||F^c\,|\psi\rangle|| \times \left(F^c\,|\psi\rangle\big/||F^c\,|\psi\rangle||\right),$$
(5)

Finally,

(i) (c) the projectors E and F commute on $|\psi\rangle$: $EF\,|\psi\rangle = FE\,|\psi\rangle$.

(ii) (a) Claim (i) (a) is valid.

(ii) (b) The analogous claim is valid for the opposite events E^c and F^c : F^c is a certain event in the state $E^c\,|\psi\rangle\big/||E^c\,|\psi\rangle||$, which is the second component in the coherent state decomposition (1).

Proof.(i) On account of idempotency of projectors, (2) implies conditions (i) (a), (b), (c). Conversely, the latter three evidently imply (2).

(ii) That (2) has (ii) (a) as its consequence has already been proved in the preceding passage, and (2) implies (ii) (b) due to the equivalence of (2) with condition (3) (cf Theorem 1 (i)). Conversely if (ii) (b) is satisfied, then $(I-E)\,|\psi\rangle = (I-F)(I-E)\,|\psi\rangle$ is valid (cf the Lemma). This gives $0 = (-F + FE)\,|\psi\rangle$. Substituting here $FE\,|\psi\rangle = E\,|\psi\rangle$, which is implied by (ii) (a), one obtains $(-F + E)\,|\psi\rangle = 0$, i. e., we are back at (2) as claimed. $\qquad\square$

Remark 1. One should note that definition (2) and conditions (i) and (ii) of Theorem 1 and condition (i) of Theorem 2 for twin events are **symmetric** in the events, whereas definition (ii) in Theorem 2 is **asymmetric** in the events, but it is symmetric in taking

the opposite event. Besides, definitions (2) and (3) are mathematical, in particular algebraic, whereas conditions (ii) of Theorem 1 and conditions (i) and (ii) of Theorem 2 are expressed in terms of entities with obvious physical meaning (with the exception of condition (i) (c) of Theorem 2, which is algebraic).

In the mentioned previous studies of twin events (and twin observables, cf the Introduction) only the algebraic definition (2) was used, and the concept was confined to composite systems. In this article we will not explore where the equivalent properties given in Theorems 1 and 2 find application. We confine ourselves to their utilization in the investigation of delayed twins that are going to be introduced in the next section.

Remark 2. It is evident in Theorems 1 and 2 that all claims are made in terms of the eigen-projectors E and E^c of the binary observable $E = 1E + 0E^c$; the eigenvalues play no role. Hence, one can take an arbitrary **binary observable** $O = o_1 E + o_2 E^c$, $o_1 \neq o_2$, and Theorems 1 and 2 are valid for it. (In Corollary 1 below we go to more general observables.)

The projectors E and F are not required to be necessarily distinct. However trivial is the claim that 'each projector is its own twin', allowing for this makes relation (2) a reflexive, symmetric and transitive one, i. e.. an equivalence relation in the set $\mathcal{P}(\mathcal{H})$ of all projectors in the state space (Hilbert space) $\mathcal{H}$ of the quantum system. One may wonder if these **equivalence classes** have a simple structure. An affirmative answer was given in the more general

case of mixed-or-pure states (density operators) in a recent study [10] (see section 5 and Theorem 5.3. there). The classes are denoted by $[E]$; it is the class to which a given projector E belongs.

Here we derive independently the equivalence classes for pure states.

Theorem 3. Let $|\psi\rangle$ be an arbitrary state vector. Any equivalence class $[E]$, i. e., the set of all projectors that are twins in $|\psi\rangle$ with an arbitrary given projector E , consists of all projectors of the form

$$E_0 + \bar{E}, \tag{6a}$$

where

$$E_0 \equiv \left(E\,|\psi\rangle\big/||E\,|\psi\rangle||\right)\left(\langle\psi|\,E\big/||E\,|\psi\rangle||\right), \ \ \text{if} \ \ E\,|\psi\rangle \neq 0; \tag{6b}$$

and

$$E_0 \equiv 0 \quad \text{if} \quad E\,|\psi\rangle = 0. \tag{6c}$$

By $\bar{E}$ is denoted any projector orthogonal to both E_0 and $|\psi\rangle\langle\psi|$.

Proof of Theorem 3. First we establish that $E_0 \in [E]$. Substituting E_0 from (6b) in $E_0\,|\psi\rangle$, we obtain

$$E_0\,|\psi\rangle = \left(E\,|\psi\rangle\big/||E\,|\psi\rangle||\right)\left(\langle\psi|\,E\big/||E\,|\psi\rangle||\right)\,|\psi\rangle = E\,|\psi\rangle,$$

because

$$\langle\psi|\,E\,|\psi\rangle = ||E\,|\psi\rangle||^2.$$

Further, it is obvious from (6b) that $EE_0 = E_0$, i. e., that E_0 is a subprojector of E : $E_0 \leq E$. It is also obvious from (6b) that E can be replaced with any other element of the class

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$[E]$ in the very definition of E_0 . Hence, E_0 is a sub-projector of every projector in $[E]$. It is the minimal element in the class.

Since Theorem 3 requires $\bar{E}$ to be orthogonal to $|\psi\rangle\langle\psi|$, it follows that $\bar{E}\,|\psi\rangle = 0$ and $(E_0 + \bar{E})\,|\psi\rangle = E_0\,|\psi\rangle$. Thus, all projectors given by (6a) belong to the equivalence class $[E]$.

Next we assume that $E' \in [E]$. Since it has been proved that $E_0 \in [E]$, and that it is the minimal element of the class, the operator $E' - E_0$ is a projector orthogonal to E_0 . Further, $(E' - E_0)\,|\,\psi\,\rangle = 0$ implies that $(E' - E_0)$ is orthogonal to $|\psi\rangle\langle\psi|$. Hence, E' is of the form (6a) with (6b), and the claims of Theorem 2 are valid.

The case $E\,|\psi\rangle = 0$ is proved analogously. □

One should notice that $E\,|\psi\rangle\big/\|E\,|\psi\rangle\|$ is the state into which $|\,\psi\,\rangle$ changes when E occurs ideally in $|\,\psi\,\rangle$ (cf Definition 1). (The former state was called the Lüders state in [10].)

Now we go to more general observables, but we confine ourselves to those with a purely discrete spectrum (allowing it to be infinite). The next definition is so formulated that the text of its part A can be utilized again below.

Definition 3. Let $|\,\psi\,\rangle$ be a state vector, $O = \sum_k o_k E_k,\ k \neq k' \ \Rightarrow\ o_k \neq o_{k'}$ and $O' = \sum_l o_l F_l,\ l \neq l' \ \Rightarrow\ o_l \neq o_{l'}$ be two observables in spectral form. They are **twin observables** or observables that are twins in $|\,\psi\,\rangle$ if

A) In both observables **all** positive-probability eigenvalues can be renumerated by an index m **common to the two observables**

so that all corresponding eigen-projectors E_m and F_m satisfy a certain condition.

B) For each value of m, the corresponding eigen-projectors E_m and F_m satisfy the condition that they are twin projectors:

$$\forall m: \quad E_m \left| \psi \right\rangle = F_m \left| \psi \right\rangle. \tag{7}$$

Definition 3 generalizes to not-necessarily-composite state vectors the definition given in [11]. In the special case when the two observables themselves act equally on the state vector, as in the illustration given in the Introduction (the singlet state), the twin observables were called algebraic ones, a special case of the physical ones.

Now we give two alternative definitions of twin observables.

Corollary 1. Two **observables** $O = \sum_k o_k E_k, \ k \neq k' \ \Rightarrow \ o_k \neq o_{k'}$ and $O' = \sum_l o_l F_l, \ l \neq l' \ \Rightarrow \ o_l \neq o_{l'}$ that are given in spectral form are **twins** in a pure state $\left| \psi \right\rangle$ if any one of the following two conditions (i) and (ii) is valid, and only if both are satisfied. The (composite, necessary and sufficient) conditions are:

(i) (a) Condition A from Definition 3 is valid.

(i) (b) All corresponding eigen-events E_m and F_m have equal probabilities in the given state:

$$\forall m: \quad \left\langle \psi \right| E_m \left| \psi \right\rangle = \left\langle \psi \right| F_m \left| \psi \right\rangle. \tag{8}$$

(i) (c) Ideal occurrence of the corresponding eigen-events gives the same state:

$$\forall m: \quad E_m \left| \psi \right\rangle \big/ \left\| E_m \left| \psi \right\rangle \right\| = F_m \left| \psi \right\rangle \big/ \left\| F_m \left| \psi \right\rangle \right\|. \tag{9}$$

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(ii) (a) Condition A from Definition 3 is valid.

(ii) (b) If any of the eigen-events E_m occurs ideally in $|\psi\rangle$, then the corresponding eigen-event F_m of the second observable becomes certain in the state that comes about, i. e., one has

$$\forall m : \quad E_m\,|\psi\rangle\big/||E_m\,|\psi\rangle|| = F_m\Big(E_m\,|\psi\rangle\big/||E_m\,|\psi\rangle||\Big) \qquad (10)$$

(cf the Lemma).

Proof follows from Theorems 1 and 2 in a straightforward way except for the sufficiency of condition (ii), which we prove as follows.

Definition 3 implies that $|\psi\rangle = \sum_{m'} E_{m'}\,|\psi\rangle$. Further, repeated use of condition (ii)(b), orthogonality and idempotency of the F_m projectors enables one to write

$$\forall m : \quad F_m\,|\psi\rangle = \sum_{m'}||E_{m'}\,|\psi\rangle|| \times F_m\Big[\Big(E_{m'}\,|\psi\rangle\big/||E_{m'}\,|\psi\rangle||\Big)\Big] =$$

$$\sum_{m'}||E_{m'}\,|\psi\rangle|| \times F_m F_{m'}\Big[\Big(E_{m'}\,|\psi\rangle\big/||E_{m'}\,|\psi\rangle||\Big)\Big] =$$

$$||E_m\,|\psi\rangle|| \times F_m\Big[\Big(E_m\,|\psi\rangle\big/||E_m\,|\psi\rangle||\Big)\Big] =$$

$$||E_m\,|\psi\rangle|| \times \Big[\Big(E_m\,|\psi\rangle\big/||E_m\,|\psi\rangle||\Big)\Big] = E_m\,|\psi\rangle.$$

$$\square$$

3 SECOND GENERALIZATION: DELAYED TWIN EVENTS

The first generalization of the notion of twin events and twin observables (in the preceding section) targeted the states of all quantum systems, not just the bipartite ones as treated in previous work. But

it all applied to a quantum pure state given at one fixed moment. In the context of delayed twins that are going to be introduced, the twins from the preceding section will be called **simultaneous twins**. Now we generalize further, allowing for unitary evolution separating the two twins and considering two moments $t \geq t_0$.

Henceforth we consider an arbitrary given pure state $|\psi, t_0\rangle$ at an arbitrary given moment t_0 . The quantum system is assumed to be isolated from its surroundings in a time interval $[t_0, t]$, $t \geq t_0$, so that the change of state is governed by a unitary evolution operator $U(t - t_0)$, which we write shortly as U . The delayed state will be interchangeably written as $U |\psi, t_0\rangle$ or as $|\psi, t\rangle$.

Definition 4. Two events E and F are **delayed twins** in the state $|\psi, t_0\rangle$ for the time interval or delay $(t - t_0)$ if

$$UE |\psi, t_0\rangle = FU |\psi, t_0\rangle \qquad (11)$$

is valid.

Remark 3. It follows from the definition given by (11) that E and $UEU^\dagger$ are delayed twins. They are **the trivial delayed twins**.

Proposition 1. Let E be an event. The equivalence class $[UEU^{-1}]$ of all events that are simultaneous twins with UEU^{-1} in $|\psi, t\rangle$ consists of all delayed events from the equivalence class $[E]$ of all simultaneous twins of E in the state $|\psi, t_0\rangle$, i. e.,

$$[UEU^{-1}] = \{UE'U^{-1} : E' \in [E]\}. \qquad (12)$$

Proof. If $E' \in [E]$, then $E' \mid \psi, t_0 \rangle = E \mid \psi, t_0 \rangle$ imply-ing $(UE'U^{-1})(U \mid \psi, t_0 \rangle) = (UEU^{-1})(U \mid \psi, t_0 \rangle)$. Hence the set on the rhs of (12) is a subset of its lhs. On the other hand, if $F \in [UEU^{-1}]$, then $F(U \mid \psi, t_0 \rangle) = (UEU^{-1})(U \mid \psi, t_0 \rangle)$. Multi-plying this from the left with U^{-1} , one obtains $(U^{-1}FU) \in [E]$. Since $F = U(U^{-1}FU)U^{-1}$, the lhs of (12) is a subset of its rhs. Hence, the two sides of the claimed relation are equal. $\square$

Remark 4. In intuitive terms, one might say that the evolution or delay splits the equivalence class $[E]$ into two equal copies; one remains unchanged as $[E]$ at t_0 , and the other, the delayed clone, appears as $[UEU^{-1}]$ at t .

Remark 5. Proposition 1 implies that the unitary evolution op-erator U from t_0 till t $\left(t \geq t_0\right)$ maps all equivalence classes into which the set of all projectors (quantum logic) $\mathcal{P}(\mathcal{H})$ is bro-ken up with respect to $\mid \psi, t_0 \rangle$ into those regarding $\mid \psi, t \rangle$.

Proposition 2. Let E, E', F, F' be four events.

A) If E, F are a pair of delayed twins in a state $\mid \psi, t_0 \rangle$ for the delay $(t - t_0)$, then so are E, F' if and only if F and F' are **simultaneous twins** in $\mid \psi, t \rangle$.

B) If the events E, F are delayed twins in $\mid \psi, t_0 \rangle$ for the delay $(t - t_0)$, then so are E', F if and only if E and E' are **simultaneous twins** in $\mid \psi, t_0 \rangle$.

Proof. A) Necessity. Utilizing relation (11), we have $UE\,|\psi,t_0\rangle = F\,|\psi,t\rangle$ and $UE\,|\psi,t_0\rangle = F'\,|\psi,t\rangle$. Hence, $F\,|\psi,t\rangle = F'\,|\psi,t\rangle$, i. e., F and F' are simultaneous twins.

Sufficiency. The relations $UE\,|\,\psi,t_0\,\rangle = F\,|\,\psi,t\,\rangle$ and $F\,|\psi,t\rangle = F'\,|\psi,t\rangle$ imply $UE\,|\psi,t_0\rangle = F'\,|\psi,t\rangle$

B) Necessity. Assuming $UE\,|\psi,t_0\rangle = F\,|\psi,t\rangle$ and $UE'\,|\psi,t_0\rangle = F\,|\psi,t\rangle$, one obtains $E\,|\psi,t_0\rangle = E'\,|\psi,t_0\rangle$.

Sufficiency. The relations $UE\,|\,\psi,t_0\,\rangle = F\,|\,\psi,t\,\rangle$ and $E\,|\psi,t_0\rangle = E'\,|\psi,t_0\rangle$ imply $UE'\,|\psi,t_0\rangle = F.\,|\psi,t\rangle$. □

Note that in Proposition 2 the equivalence classes $[E]$ and $[UEU^{-1}]$ play symmetrical roles with respect to delay and anti-delay or inverse delay.

Theorem 4. A) Let E be an arbitrary event and let E' be an arbitrary event from the equivalence class $[E]$ of all simultaneous twins with E in $|\,\psi,t_0\,\rangle$. Let F also be an event. The events E' and F are delayed twins for a given time interval $(t - t_0)$ if and only if F is a simultaneous twin with the event $UEU^{\dagger}$ in $|\,\psi,t\,\rangle$ $\left(\equiv U\,|\psi,t_0\rangle \right)$.

B) Symmetrically: Let F be an arbitrary event and F' an arbitrary event from the equivalence class $[F]$ of all simultaneous twins with F in $|\,\psi,t\,\rangle$. Let E also be an event. The events E and F' are delayed twins for $(t - t_0)$ if and only if E is a simultaneous twin with the event $U^{\dagger}FU$ in $|\,\psi,t_0\,\rangle$.

Proof. A) It follows from Proposition 2(A) that E' and UEU^{-1} are delayed twins. The former is a delayed twin with $UE'U^{-1}$, and

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this is, in turn, a simultaneous twin with UEU^{-1} in the later state. Then Proposition 2(B) implies that E' and F are delayed twins if and only if the latter is a simultaneous twin with $UE'U^{-1}$, and, due to transitivity, if and only if F is a simultaneous twin with UEU^{-1} in the later state.

B) is proved analogously. □

Note that also in Theorem 4 the equivalence classes $[E]$ and $[UEU^{-1}]$ play symmetrical roles with respect to delay and antidelay or inverse delay.

Definition 5. Two pairs of delayed twin events E, F and E', F' in the same pure state $|\psi, t_0\rangle$ and for the same delay $(t - t_0)$ are **equivalent** if E and E' are simultaneous twins in the earlier state $|\psi, t_0\rangle$, and so are F and F' in the later state $|\psi, t\rangle$.

Two Remarks, obvious consequences of Definition 5, and an additional Remark help to fully understand the concept of equivalent pairs of delayed twin events.

Remark 6. Two pairs of delayed twin events E, F and E', F' in the same state $|\psi, t_0\rangle$ for the same time interval $(t-t_0)$ are **inequivalent** if and only if **both** E and E' are **not** simultaneous twins in $|\psi, t_0\rangle$ and F and F' are are **not** simultaneous twins in $|\psi, t\rangle$.

Remark 7. It cannot happen that two pairs of delayed twin events E, F and E', F' are inequivalent with respect to $|\psi, t_0\rangle$ and for $(t-t_0)$ because E and E' fail to be simultaneous twins in $|\psi, t_0\rangle$ though F and F' are

simultaneous twins in $|\psi, t\rangle$ and symmetrically with respect to delay-antidelay.

Remark 8. The claims of Theorem 4, Corollary 2 and Remark 6 amount to saying that there is no cross-twinning, i. e., twinning across the equivalence classes. This is understandable intuitively in view of Remark 4 and Remark 5.

Now, following the claims of Theorem 2 from the preceding section, we give four necessary and sufficient conditions, or equivalent definitions, for **delayed twin events**, valid when the given projector E does not nullify the given state $|\psi, t_0\rangle$.

Theorem 5. Let $E\,|\psi, t_0\rangle \neq 0$. Two events E and F are **delayed twins** in $|\psi, t_0\rangle$ for $(t - t_0)$ (cf Definition 4) if any of the following four conditions is valid, and only if all four are. The (simple or composite) conditions (i)-(iv) read:

(i) The opposite events E^c and F^c are delayed twins in $|\psi, t_0\rangle$:

$$UE^c\,|\psi, t_0\rangle = F^c U\,|\psi, t_0\rangle. \tag{13}$$

(ii) (a) The event E has the **same probability** in the state $|\psi, t_0\rangle$ as the event F in the delayed state $|\psi, t\rangle$:

$$\langle\psi, t_0|\,E\,|\psi, t_0\rangle = \langle\psi, t|\,F\,|\psi, t\rangle. \tag{14a}$$

(ii) (b) The changed state due to ideal occurrence of E in the state $|\psi, t_0\rangle$ is, after evolution, equal to the changed state brought about by ideal occurrence of the event F in the delayed state $|\psi, t\rangle$:

$$U\Big(E\,|\psi, t_0\rangle\Big/||E\,|\psi, t_0\rangle||\Big) = F\,|\psi, t\rangle\Big/||F\,|\psi, t\rangle|| \tag{14b}.$$

(Intuitively put: Collapse and evolution commute.)

(iii) (a) When the state $E\,|\psi, t_0\rangle\Big/||E\,|\psi, t_0\rangle||$, which is a component in the coherent state decomposition like (1), is delayed by $U\,\Big(\equiv U(t - t_0)\Big)$, F is a certain event in the delayed state:

$$U\Big(E\,|\psi, t_0\rangle\Big/||E\,|\psi, t_0\rangle||\Big) = F\Big[U\Big(E\,|\psi, t_0\rangle\Big/||E\,|\psi, t_0\rangle||\Big)\Big]. \tag{15a}$$

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(iii) (b) When the state $F \mid \psi, t\rangle \big/ \|F \mid \psi, t\rangle\|$, which is a component in the corresponding coherent decomposition of the state $\mid \psi, t\rangle$, is inversely delayed, then the event E is **certain** in the past, inversely delayed state $U^{-1}\left(F \mid \psi, t\rangle \big/ \|F \mid \psi, t\rangle\|\right)$:

$$U^{-1}\left(F \mid \psi, t\rangle \big/ \|F \mid \psi, t\rangle\|\right) = E\left[U^{-1}\left(F \mid \psi, t\rangle \big/ \|F \mid \psi, t\rangle\|\right)\right]. \qquad (15b)$$

(iv) (a) If the event E occurs ideally in the state $\mid \psi, t_0\rangle$, then the event F is **certain** in its corresponding delayed state $U\left(E \mid \psi, t_0\rangle \big/ \|E \mid \psi, t_0\rangle\|\right)$.

(iv) (b) The analogous statement is valid for the opposite events E^c and F^c respectively.

Proof consists in the simple fact that each of the four conditions is valid if and only if the corresponding condition in Theorem 1 is valid for the events $UEU^\dagger$ and F , making them simultaneous twins in the state $\mid \psi, t\rangle$ $\big(\equiv U \mid \psi, t_0\rangle\big)$. To make this transparent, we write out the details item by item.

(i) It is straightforward to see that relation (12) can be rewritten as

$$\left(UE^cU^\dagger\right) \mid \psi, t\rangle = F^c \mid \psi, t\rangle$$

(cf (2) in Theorem 1).

(ii) Relation (14a) can be written in the form

$$\langle \psi, t \mid \left(UEU^\dagger\right) \mid \psi, t\rangle = \langle \psi, t \mid F \mid \psi, t\rangle.$$

Further, relation (14b) is equivalent to

$$\left(UEU^\dagger\right) \mid \psi, t\rangle \big/ \|(UEU^\dagger) \mid \psi, t\rangle\| = F \mid \psi, t\rangle \big/ \|F \mid \psi, t\rangle\|$$

(cf Theorem 1 (ii)).

(iii) Relation (15a) can take the form

$$\left(UEU^{-1}\right) \mid \psi, t\rangle \big/ \|(UEU^{-1}) \mid \psi, t\rangle\| = F\left[\left(UEU^{-1}\right) \mid \psi, t\rangle \big/ \|(UEU^{-1}) \mid \psi, t\rangle\|\right]$$

(cf Theorem 1 (iii) (a)). Further, relation (15b) is equivalent to

$$F \mid \psi, t\rangle \big/ \|F \mid \psi, t\rangle\| = \left(UEU^\dagger\right)\left(F \mid \psi, t\rangle \big/ \|F \mid \psi, t\rangle\|\right)$$

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(cf (iii)(b) in Theorem 1).

(iv) Claim (a) is proved above in (iii) (a). Claim (b) is proved analogously.
□

4 TWIN OBSERVABLES

Now we go to observables with a purely discrete spectrum (allowing it to be infinite). The next definition is written in a redundant way, just like the analogous definition in the preceding section, in order to make possible repeated use of its condition A.

Definition 6. Let $O = \sum_k o_k E_k$, $k \neq k' \Rightarrow o_k \neq o_{k'}$ and $O' = \sum_l o_l F_l$, $l \neq l' \Rightarrow o_l \neq o_{l'}$ be two observables in spectral form, and $|\psi, t_0\rangle$ a state vector. One is dealing with **delayed twin observables** or observables that are delayed twins in the state $|\psi, t_0\rangle$ for the time interval or delay $(t - t_0)$ if

A) All of the positive-probability eigenvalues of O and O' in $|\psi, t_0\rangle$ can be renumerated by a common index m so that the corresponding eigen-projectors E_m and F_m satisfy a given condition.

B) Assuming validity of condition A, the corresponding eigen-projectors E_m and F_m satisfy the condition that they are delayed twin projectors for each value of m :

$$\forall m : \quad U E_m |\psi, t_0\rangle = F_m |\psi, t\rangle, \tag{16}$$

where $U \equiv U(t - t_0)$.

Next we give two alternative definitions of twin observables, which are potentially important for applications.

Theorem 6. Two observables $O = \sum_k o_k E_k$, $k \neq k' \Rightarrow o_k \neq o_{k'}$ and $O' = \sum_l o_l F_l$, $l \neq l' \Rightarrow o_l \neq o_{l'}$ (given in spectral form) are **delayed twins** in a pure state $|\psi, t_0\rangle$ for $(t - t_0)$ if any one of the following two

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(composite) conditions (i) and (ii) is valid, and only if both are satisfied. The conditions are:

(i) (a) Condition A in Definition 6 is satisfied.

(i) (b) The corresponding eigen-events have equal probabilities in the respective states:

$$\forall m : \quad \langle \psi, t_0 | \, E_m \, | \psi, t_0 \rangle = \langle \psi, t | \, F_m \, | \psi, t \rangle. \tag{17}$$

(i) (c) ideal occurrence of the corresponding eigen-events gives the same state up to the delay:

$$\forall m : \quad U \Big(E_m \, | \psi, t_0 \rangle \big/ ||E_m \, | \psi, t_0 \rangle|| \Big) = F_m \, | \psi, t \rangle \big/ ||F_m \, | \psi, t \rangle||. \tag{18}$$

(ii) (a) Condition A in Definition 6 is valid.

(ii) (b) If any of the eigen-events E_m occurs in ideal measurement in $| \psi, t_0 \rangle$, then the corresponding eigen-event F_m of the second observable becomes certain in the state that comes about in the measurement and becomes delayed:

$$\forall m : \quad U \Big(E_m \, | \psi, t_0 \rangle \big/ ||E_m \, | \psi, t_0 \rangle|| \Big) =$$
$$F_m \Big[U \Big(E_m \, | \psi, t_0 \rangle \big/ ||E_m \, | \psi, t_0 \rangle|| \Big) \Big]. \tag{19}$$

Proof (i) This claim coincides with that of Theorem 4, condition (ii). Hence, it has already been proved that this condition is necessary and sufficient for the observables being delayed twins.

(ii). Necessity is obvious from Theorem 4 (iv).

Sufficiency. Repeated use of condition (ii), orthogonality and idempotency of the F_m projectors, and the fact that

$$\sum_{m'} E_{m'} \, | \psi, t_0 \rangle = | \psi, t_0 \rangle,$$

enable one to write

$$\forall m : \quad F_m U \Big[\, | \psi, t_0 \rangle \Big] = \sum_{m'} ||E_{m'} \, | \psi, t_0 \rangle|| \times F_m U \Big[\Big(E_{m'} \, | \psi, t_0 \rangle \big/ ||E_{m'} \, | \psi, t_0 \rangle|| \Big) \Big] =$$

$$\sum_{m'} ||E_{m'} |\psi, t_0\rangle|| \times F_m F_{m'} U \left[\left(E_{m'} |\psi, t_0\rangle \Big/ ||E_{m'} |\psi, t_0\rangle|| \right) \right] =$$

$$||E_m |\psi, t_0\rangle|| \times F_m U \left[\left(E_m |\psi\rangle \Big/ ||E_m |\psi, t_0\rangle|| \right) \right] =$$

$$||E_m |\psi, t_0\rangle|| \times U \left[\left(E_m |\psi, t_0\rangle \Big/ ||E_m |\psi, t_0\rangle|| \right) \right] = U E_m |\psi, t_0\rangle.$$

$$\square$$

Claim (i) (c) of Theorem 6, which is the corner stone of the next theorem, can be put intuitively and roughly as follows: - Selective collapse and evolution commute for delayed twin observables.-

Theorem 6 (i) has an obvious consequence that may be very important for applications (see e. g. subsection 5.2 below). Hence it must be spelled out though its precise formulation is cumbersome. To emphasize its expected importance, we write it as a theorem. In intuitive and imprecise but concise terms we can put its claim as: - Nonselective collapse commutes with evolution for delayed twin observables.-

Theorem 7. Let two observables O and O' be delayed twins for a given state $|\psi, t_0\rangle$ and for a given interval $(t - t_0)$ as in Theorem 6. We compare two situations.

A) Nonselective ideal measurement of O, i. e. its ideal measurement on an entire ensemble of quantum systems, in the state $|\psi, t_0\rangle$ is performed at t_0 (thus all positive-probability results appear on some systems in the ensemble). The measurement converts the pure state into the **mixture**

$$\rho_{t_0} = \sum_m w_m^0 \rho_m^0, \qquad (20a)$$

$$\forall m: \ \rho_m^0 \equiv \left(E_m |\psi, t_0\rangle \Big/ ||E_m |\psi, t_0\rangle|| \right) \left(\langle \psi, t_0| E_m \Big/ ||E_m |\psi, t_0\rangle|| \right), \quad (20b)$$

$$\forall m: \quad w_m^0 \equiv \langle \psi, t_0| E_m |\psi, t_0\rangle. \qquad (20c)$$

B) As an alternative, we take the situation when the pure state $|\psi, t_0\rangle$ evolves unitarily till the moment t, becoming $|\psi, t\rangle \equiv U |\psi, t_0\rangle$, and

then ideal nonselective measurement of the **delayed twin observable** O' is carried out on the ensemble described by $|\psi, t\rangle$ resulting in the mixed state

$$\rho_t \equiv \sum_m w_m^t \rho_m^t, \tag{21a}$$

$$\forall m: \ \rho_m^t \equiv \left(F_m\,|\psi, t\rangle \Big/ \|F_m\,|\psi, t\rangle\|\right)\left(\langle\psi, t|\,F_m \Big/ \|F_m\,|\psi, t\rangle\|\right), \tag{21b}$$

$$\forall m: \quad w_m^t \equiv \langle\psi, t|\,F_m\,|\psi, t\rangle. \tag{21c}$$

It is **claimed** that:

C) The unitarily evolved mixed state $U\rho_{t_0}U^{-1}$ and the state ρ_t are **equal**:

$$U\rho_{t_0}U^{-1} = \rho_t. \tag{22}$$

D) The statistical weights w_m^0 of the pure states ρ_m^0, which are also the probabilities of the individual results o_m in the measurement of O in $|\psi, t_0\rangle$, are **equal** to the statistical weights w_m^t of the corresponding pure states ρ_m^t, which are also the probabilities of the results o_m' of the measurement of the delayed twin observable O' in $|\psi, t\rangle$:

$$\forall m: \quad \langle\psi, t_0|\,E_m\,|\psi, t_0\rangle = w_m^0 \ = \ w_m^t = \langle\psi, t|\,F_m\,|\psi, t\rangle. \tag{23}$$

Corollary 2. Delayed twin observables have the **chaining property** in the following sense. If two observables $O = \sum_k o_k E_k, \ k \neq k' \ \Rightarrow \ o_k \neq o_{k'}$ and $O' = \sum_l o_l F_l, \ l \neq l' \ \Rightarrow \ o_l \neq o_{l'}$ (given in spectral form) are delayed twins in the pure state $|\psi, t_0\rangle$ for the time interval $(t - t_0)$, and if the latter observable and a third observable $O'' = \sum_n o_n'' E_n, \ n \neq n' \ \Rightarrow \ o_n'' \neq o_{n'}''$ are delayed twins in the state $|\psi, t\rangle \ \left(\equiv U(t - t_0)\,|\psi, t_0\rangle\right)$ for $\left((t + t') - t\right)$, then the first observable and the third one are delayed twins in the state $|\psi, t_0\rangle$ for $\left((t + t') - t_0\right)$.

Proof is obvious.

Remark 9. The notion of a chain of delayed twin observables comes close to the well known von Neumann chain (see e. g. [12]) and the infinite regress

to which it leads (cf e. g. [13]). Since the former is based on the concept of delayed twins, and they are well understood in terms of the preceding results, the former chain may perhaps contribute to a better understanding of the latter.

Remark 10. The chaining property of delayed twins may come close also to the concept of Consistent Histories, which was extensively studied by Griffiths [14], Gell-Mann and Hartle [15] and [16] (who called it 'decoherent histories'), as well as by Omnes [17]. This concept has the ambition to give a new interpretation of quantum mechanics . (It will not be further discussed in this article.)

The delayed-twin concept can find many applications in quantum mechanics as will be shown in follow-ups. In the next and last section we sketch a few applications.

5   APPLICATIONS

Now we describe shortly a few cases where the delayed-twin-observables concept can be seen to appear. By this we do not care whether it is useful or not in the mentioned example.

5.1   Some Preparations of Quantum Experiments

Let us take the concrete example of preparing a 1/2-spin particle in spin-up state by the Stern-Gerlach measuring instrument adapted for preparation. To this purpose, there is open space instead of the upper part of the screen (or instead of an upper detector). Let event E for the particle be "passage through the upper part" at t_0 , and let event F for it be measurement at t, $t > t_0$, to the right of the described preparator. Imagine that the geometry is such that F occurs if and only if so does E in the previous moment (when the preparation took successfully place). The two events are delayed twins according to condition (or definition) (iv) in Theorem 5.

One should not be confused by the fact that the event E does not actually 'occur' in the quantum-mechanical sense in the described preparation because it is not observed (measured). Theoretically one can assume that it does because ideal selective position measurement of E does not change the up-going component in the Stern-Gerlach instrument. It is, of course, simpler to view just the collapse caused by the ideal occurrence of F at the later moment of actual measurement. But, as Bohr would say, "visualization" (classical intuition) does not allow us to understand the occurrence of F unless E has occurred (though unobserved) at t_0 .

Whatever kind of measurement is performed at t , we can again insert position collapse to the spatial domain occupied by the measuring instrument immediately before the actual measurement takes place because it does not change the component of the state of the particle at t , which (locally) interacts with the instrument.

5.2 A Puzzle in Understanding the Real Scully et al. Experiment

Scully et al. have performed a very sophisticatedly upgraded two-slit interference-or-which-way experiment [22]. Instead of 'passing two slits', two excited atoms undergo cascade de excitation coherently emitting a pair of photons in opposite directions. Detectors are placed to the right of the atoms to detect the right-going photon, and detectors are put to the left to detect the photon moving to the left. The experimental arrangement is such that the photon that moves to the right is detected before the left-moving one reaches a detector.

Without going into details of the intricate experiment, theoretical analysis of the experiment led to a puzzle. Namely, Scully et al. have shown [23] that one can view the experiment so that the right-moving photon undergoes spatial collapse at (various positions of) the corresponding detector, then one evaluates the effect of this collapse on the left-moving photon on account of the entanglement between the two photons, and at the end one evaluates the relative frequencies of the coincidence detections.

This first-quantization description of Scully et al. is precisely what one would expect in the Copenhagen approach. The puzzle was due to the fact that the present author gave a successful description of the same experiment [22] without collapse of the right-moving photon [24]. Which of the two mutually contradicting pictures is the correct description of the experiment?

The puzzle is immediately solved by Theorem 7. If t_0 is the moment of collapse of the right-moving photon, observation of its detection at a certain location of the right-placed detector can be viewed as ideal measurement (by the sentient observer) of an observable O, and the corresponding observation of the same detection result at moment t, when the left-moving photon has reached a detector, can be viewed as ideal measurement of a delayed twin observable O' because a reading at t can be done if and only if the same result could (or could have been) done at t_0. Then the commutation of collapse and evolution, with equality of the probabilities, claimed in Theorem 7 explains the puzzle.

As my late teacher professor R. E. Peierls used to say, "paradoxes are useful because one can learn from their solutions". Here we learn that quantum experiments can be viewed not only in different, but also in different mutually exclusive ways. Each view provides us with some specific advantage.

5.3 Exact Quantum Measurement

Let the object on which the measurement is performed be subsystem A, and let the measuring instrument be subsystem B. Further, let $O_A = \sum_k o_k E_A^k$. $k \neq k' \Rightarrow o_k \neq o_{k'}$ be the observable that is measured, and let $P_B = \sum_k o'_k F_B^k$. $k \neq k' \Rightarrow o'_k \neq o'_{k'}$ be the so-called 'pointer observable' (both given in spectral form). The measuring instrument 'takes cognizance' of a result o_k (or rather of the occurrence of the corresponding eigen-event E_A^k) in terms of the 'pointer position' o'_k (actually by the occurrence of F_B^k) that corresponds to it. (Note the common index.)

Exact measurement (henceforth only 'measurement') is defined in quantum mechanics by the **calibration condition** [18], [19]: If an event E_A^k is certain

(has occurred) in the initial state $\mid \psi, t_0 \rangle_A$ of the object, then the corresponding event F_B^k on the measuring instrument is certain (occurs) in the final composite-system state $\mid \Psi, t\rangle_{AB}^f \equiv U_{AB}\Big(\mid \psi, t_0\rangle_A \mid \psi, t_0\rangle_B^i \Big)$ of the object-plus-instrument system after the measurement interaction has ended. Here $U_{AB} \equiv U_{AB}(t_f - t_i)$ is the unitary operator that contains the measurement interaction; $\mid \psi, t_0\rangle_B^i$ is the initial or 'trigger-ready' state of the measuring instrument; and the indices f and i refer to the final and the initial moments respectively.

In Bohrian collapse-interpretations of quantum mechanics one says that the operator U_{AB} describes premeasurement (measurement short of collapse) [18]; in no-collapse, relative-state interpretations it is the entire dynamical law of measurement[20], [21] . We shall write '(pre)measurement', to keep an open mind about the interpretation.

In algebraic form this reads

$$|\psi, t_0\rangle_A = E_A^k\, |\psi, t_0\rangle_A \quad \Rightarrow \quad |\Psi, t\rangle_{AB}^f = F_B^k\, |\Psi, t\rangle_{AB}^f \qquad (24)$$

(cf the Lemma).

An arbitrary initial state $\mid \psi, t_0\rangle_A \mid \psi, t_0\rangle_B^0$ can be decomposed into its sharp measured-observable eigenvalue components

$$|\psi, t_0\rangle_A\, |\psi, t_0\rangle_B^0 = \sum_k ||E_A^k\, |\psi, t_0\rangle_A|| \times \Big((|\psi, t_0\rangle_A \big/ ||E_A^k\, |\psi, t_0\rangle_A||)\, |\psi, t_0\rangle_B^0 \Big).$$

The components are precisely those states into which selective ideal measurement of the individual values o_k of the measured observable would turn the initial state. On the other hand, the calibration condition, and the fact that the evolution operator is linear, imply that U would evolve each component into a final state with the corresponding sharp value of the pointer observable.

According to Theorem 6 condition (ii), the (pre)measurement evolution makes the measured observable and the pointer observable delayed twins in the initial state for the interval $(t_f - t_i)$.

At first glance one might jump to the conclusion that every case of opposite-subsystem delayed twin observables in a bipartite state can be viewed as measurement. The fallacy is in the fact that measurement, being defined in terms

of the calibration condition, makes the measured observable and the pointer observable twins in **all** initial states of the measured object A. But still hopefully the delayed twin concept for opposite-subsystem observables is the backbone of the notion of measurement in quantum mechanics.

The measured observable and the pointer one , being twins, 'inherit' the nice properties of twins. No doubt, the most important one is the so-called **probability reproducibility condition** [18], [19] (cf Theorem 5 (i) (b) above). It says that the probability of a result o_k in the initial state $|\psi, t_0\rangle_A |\psi, t_0\rangle_B^0$ **equals** the probability of the corresponding pointer position o'_k in the final state:

$$\forall k: \quad \langle \psi, t_0 |_A E_A^k |\psi, t_0\rangle_A = \langle \Psi, t |_{AB} F_B^k |\Psi, t\rangle_{AB}. \tag{25}$$

The probability reproducibility condition is crucial for measurement in ensembles, because it enables the statistical information to become transferred from the ensemble of quantum objects to the ensemble of results (relative frequencies) on the measuring instrument.

Foundationally-minded physicists prefer nondemolition (synonyms: repeatable, first kind) measurement, in which the value o_k of the measured observable, if sharp in the initial state $|\psi, t_0\rangle_A$, remains preserved (non demolished); hence it can be checked in an immediately repeated measurement. Unfortunately, numerous laboratory measurements are of the opposite, demolition type. (One should only be reminded of experiments with photons, in which the photon becomes absorbed in a detector at the end.)

One wonders about the meaning of the result in demolition measurements when the results cannot be checked. The delayed-twin concept might be helpful here. The very definition $\forall k: \left(U_{AB} E_a^k \right) \left(|\psi\rangle_A |\psi\rangle_B^0 \right) = F_B^k U_{AB} \left(|\psi\rangle_A |\psi\rangle_B^0 \right)$ of delayed twins can be slightly changed into the form

$$\forall k: \quad U_{AB}\left[\left(E_a^k |\psi\rangle_A \Big/ ||E_A^k |\psi\rangle_A|| \right) |\psi\rangle_B^0 \right] = F_B^k |\Psi\rangle_{AB}^f \Big/ ||F_B^k |\Psi\rangle_{AB}^f|| \tag{26}$$

because $||E_A^k |\psi\rangle_A|| = ||F_B^k |\Psi\rangle_{AB}^f||$ is the square root of the probability (cf (21) (b)).

Relation (26) tells us that the part of the final state that corresponds to a sharp result o'_k of the pointer observable is actually the time-evolved part of

the initial state with a sharp value of the measured observable.

In Copenhagen-inspired collapse interpretations of quantum mechanics, the usefulness of relation (26) is questionable because it suggests that collapse of the final state to a definite result is tightly bound with retroactive collapse at the initial moment [19], [25].

The relation is more useful in Everettian relative-state interpretations, where 'collapse' is replaced by 'belonging to a branch'. Then, what (26) says is that the entire process of measurement with the initial and the final moment belong to the same branch.

Even in nondemolition measurement one has trouble answering the question "Does measurement create or find the result?" Relation (26) suggests the latter answer in some sense.

Finally, the chaining property of delayed twin observables (cf Corollary 2) enables the measurement results to be transferred to another system. For instance, an observable O_A is measured in terms of a pointer observable P_B (delayed twins). Then a human observer (not distinct from another measuring instrument quantum-mechanically) takes cognizance of the results in terms of his consciousness contents, which is, e. g., observable O'. This experimenter can communicate the results (I use the plural because I have an ensemble measurement in mind) to another human being, or write them down etc. This would be another link in the chain in terms of a fourth observable that is a delayed twin with the third.

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